Further Pure Mathematics 2 Practice Paper 1 – answers

Exam-style practice: A level

1 $x \equiv 31 \pmod{75}$

2 a 56

b Cayley table is

$\times_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

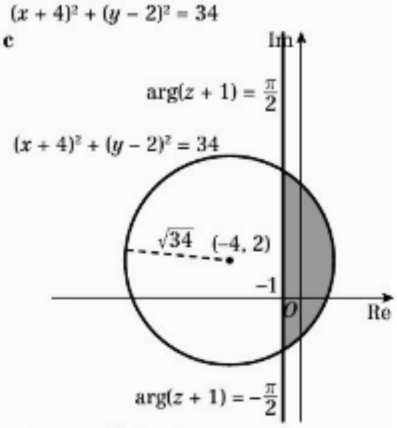
Closure: All entries in the Cayley table are in S_A.
Identity: The row and column corresponding to 1 are the same as the column and row headings, so 1 is the identity.
Inverse: All elements are self-inverse
Associativity: Assumed
So S_A forms a group under $\times_{12}$.
Since all elements have order $\leqslant 2$, there are no elements that can act as generator for the group, so S_A is a non-cyclic group.

c S_B has element 3 with order 4, so S_B is a cyclic group of order 4. S_C has $1^2 = 3^2 = 5^2 = 7^2 = 1$, so has no elements of order 4, so $S_C \not\cong S_B$. Since there are only two possible groups of order 4, S_A must be isomorphic to either S_B or S_C.

d Assume $n \geqslant 6$. Then $2^2 = 4$, which is not in the set, so the set is not closed under $\times_n$, so cannot be a group. So $n \leqslant 4$.
When n is either 2 or 4, $2^2 = 4 \equiv 0$, but 0 is not in the set either, so the set is not closed under $\times_n$. Therefore the set cannot form a group under $\times_n$ for any even n.

3 a $(x+4)^2 + (y-2)^2 = 34$

b, c



d $-1 + 7\mathrm{i}$ and $-1 - 3\mathrm{i}$

4 a i $\sqrt{2}$

ii 2; 2 is repeated as $(\lambda - 2)$ is a repeated factor in the characteristic equation.

b $\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

c $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$

5 a $I_{n+2} = S_{n+2} + M_{n+2} + D_{n+2}$
$= \frac{1}{6}I_{n+1} + \left(\frac{2}{3}I_{n+1} - \frac{1}{6}I_n\right) + d$
$= \frac{5}{6}I_{n+1} - \frac{1}{6}I_n + d$

b $I_n = 5d\left(\frac{1}{3}\right)^n - 7d\left(\frac{1}{2}\right)^n + 3d$

c As $n \to \infty$, $I_n \to 3d$

6 a $a = 2$ **b** $\frac{16\pi}{3}$

7 a $I_{n+1} = \left[-\cos x \sin^{2n+1} x\right]_0^\pi + (2n+1)\int_0^\pi \sin^{2n} x \cos^2 x \, dx$
$= (2n+1)\int_0^\pi \sin^{2n} x (1 - \sin^2 x) \, dx$
$= (2n+1)(I_n - I_{n+1})$
$\Rightarrow I_{n+1} = \frac{2n+1}{2n+2} I_n$

b Basis: $n = 0$: $\frac{0! \times \pi}{(0!)^2 \times 2^0} = \pi$
Assumption: $\int_0^\pi \sin^{2k} x \, dx = \frac{(2k)!\pi}{(k!)^2 2^{2k}}$
Induction:
$\int_0^\pi \sin^{2(k+1)} x \, dx = I_{k+1} = \frac{2k+1}{2k+2}\int_0^\pi \sin^{2k} x \, dx$
$= \frac{(2k+1)(2k)!\pi}{2(k+1)(k!)^2 2^{2k}} = \frac{(2k+2)!\pi}{2^2(k+1)^2(k!)^2 2^{2k}}$
$= \frac{(2(k+1))!\pi}{((k+1)!)^2 2^{2(k+1)}}$
So if the solution is valid for $n = k$, it is valid for $n = k + 1$
Conclusion: The solution is valid for all $n \in \mathbb{Z}$, $n \geqslant 0$.

8 a 2916 **b** 3439